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COMPARISON OF HUBER-MISES AND TRESKA YIELD CRITERIA

The purpose of the paper is to compare the Huber-Mises and Treska yield criteria. The paper has a review character. The Huber-Mises and Treska yield criteria are the most often used in engineering practice. The literature on various forms of yield conditions is broad (see [1-25], for instance). In isotropic material the loading function F involves the principal components of symmetric stress tensor $\boldsymbol{\sigma}$, i.e. the three principal stresses σ_1, σ_2 and σ_3. The principal stresses can be expressed in terms of the three first invariants of the stress tensor. We denote the first invariant of the stress tensor by $J_{1\sigma}$, the second invariant of the stress deviator tensor $\mathbf{s} = \boldsymbol{\sigma} - (\mathrm{tr}\,\boldsymbol{\sigma}/3)\mathbf{1}$ by J_{2s} and the third invariant of the stress deviator tensor by J_{3s}

$$J_{1\sigma}(\boldsymbol{\sigma}) = \mathrm{tr}\,\boldsymbol{\sigma} = 3p \tag{1}$$

$$h^2 = J_{2s}(\boldsymbol{\sigma}) = \frac{1}{2} s_{ij} s_{ji} = \frac{1}{2}\mathrm{tr}(\mathbf{s}\cdot\mathbf{s}) \tag{2}$$

$$J_{3s}(\boldsymbol{\sigma}) = \frac{1}{3} s_{ij} s_{jk} s_{kl} \tag{3}$$

By (2) we have

$$h^2 = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right] \tag{4}$$

For isotropic models of plasticity the loading function can be represented by

$$F = F(h, p) \tag{5}$$

Consider the space $\{\sigma_i\}$ with **0** as the origin. Let the point **1** $(\sigma_1, \sigma_2, \sigma_3)$ represents the stress state $\boldsymbol{\sigma}$. Let the point **2** is its orthogonal projection with regard to the Euclidean product onto trisector (Δ) defined by the unit vector with $\left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$ as cosine directions. The distances $\overline{\mathbf{02}}$ and $\overline{\mathbf{12}}$ can by expressed as

$$\overline{\mathbf{02}} = \sqrt{3}\sigma \qquad \overline{12}^2 = \overline{01}^2 - \overline{02}^2 = 2h^2 \tag{6}$$

In the loading point space $\{\sigma_i\}$ the yield surface defined by F = 0 is the axisym- me- tric surface around the trisector (Δ) as illustrated in Figure 1.

The loading function F in the case of isotropic hardening materials is expressed in the form

$$F = F\,(h, p, \boldsymbol{\eta}) \tag{7}$$

where $\boldsymbol{\eta}$ is the hardening force describing the evolution of the yield surface in loading point space $\{\sigma_i\}$. In isotropic hardening the yield surface is derived through a homothety of center **0** in the loading point space $\{\sigma_i\}$. Then the hardening force $\boldsymbol{\eta}$ reduces to a scalar variable η which defines this homothety.

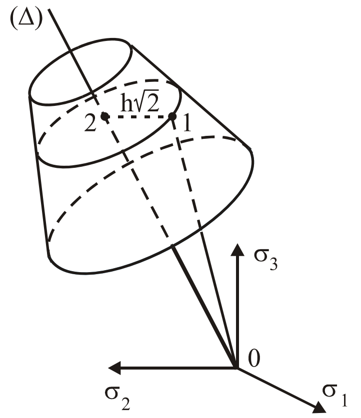

Fig. 1. Isotropic criteria of plasticity in the space $\{\sigma_i\}$

The expression (7) can be written as

$$F = F\,(h, p, \eta) \tag{8}$$

The loading function given by (8) can be expressed as a homogeneous polynome of degree n with regard to h and η

$$F = F\,(h, p, \eta) = \eta^n\, F\,(h/\eta, p/\eta, 1) \tag{9}$$

where by convection η is specified as the ratio of the homothety that transforms the yield surface defined by η = 1 into the present yield surface. In kinematic hardening, the yield surfaces are defined from each other through a translation in the loading point space $\{\sigma_i\}$. The hardening force $\boldsymbol{\eta}$ reduces to a second-order symmetric tensor $\boldsymbol{\eta}$ that defines this translation

$$F = F\,[J_{2s}(\boldsymbol{\sigma} + \boldsymbol{\eta}), J_{1\sigma}(\boldsymbol{\sigma} + \boldsymbol{\eta})] \tag{10}$$

In space $\{\sigma_i\}$ vector (η) represents the vector of translation that transforms the yield surface defined by $(\eta) = (0)$ into the present yield surface.
Assume the convex loading function for the isotropic plastic material

$$F(h, p) = h + \alpha p - q \tag{11}$$

where α and q are material characteristics. The constant q is necessarily non-negative to ensure that the zero loading point satisfies $F(0,0) \le 0$. The coefficient α is non-negative to describe an infinite tensile stress. The yield surface given by (11) is an axisymmetric surface around the trisector in principal stress space $\{\sigma_i\}$. If $\alpha = 0$ the loading function reduces to the Huber-Mises loading function.

The form of the Huber-Mises loading function is of the form

$$F = \frac{1}{\sqrt{3}}\sqrt{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11}\sigma_{22} - \sigma_{22} - \sigma_{33} - \sigma_{33}\sigma_{11} + 3\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)} - q \tag{12}$$

and for principal directions

$$F = \frac{1}{\sqrt{3}}\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1} - q \tag{13}$$

The equivalent form of the Huber-Mises loading function is

$$F = \frac{1}{\sqrt{6}}\sqrt{\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - \sigma_{11}\right)^2 + 6\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)} - q \tag{14}$$

or

$$F = \frac{1}{\sqrt{6}}\sqrt{\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2} - q \tag{15}$$

The Huber-Mises loading function can be transformed to the equivalent forms if we introduce material parameter $q = {}^{1}\!/_{\sqrt{3}}\,\sigma_o$, where σo is the yield point of the material in uniaxial tension. Then the Huber-Mises loading function is expressed in the frequently met form:

$$F = \frac{1}{\sqrt{3}}\sqrt{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11}\sigma_{22} - \sigma_{22} - \sigma_{33} - \sigma_{33}\sigma_{11} + 3\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)} - \frac{1}{\sqrt{3}}\sigma_o \tag{16}$$

$$F = \frac{1}{\sqrt{3}}\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1} - \frac{1}{\sqrt{3}}\sigma_o \tag{17}$$

$$F = \frac{1}{\sqrt{6}}\sqrt{\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - \sigma_{11}\right)^2 + 6\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)} - \frac{1}{\sqrt{3}}\sigma_o \tag{18}$$

or

$$F = \frac{1}{\sqrt{6}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} - \frac{1}{\sqrt{3}}\sigma_o \quad (19)$$

In order to present its geometrical interpretation, the Huber-Mises criterion is rewritten using principal stress deviator components as

$$F = \frac{1}{\sqrt{2}}\sqrt{s_1^2 + s_2^2 + s_3^2} - q = 0 \quad (20)$$

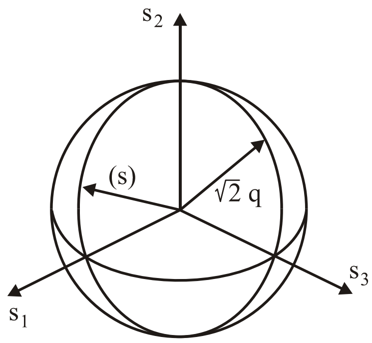

Fig. 2. The Huber-Mises yield locus in the space $\{s_i\}$ of principal stress deviators

In the space $\{s_i\}$ the expression (19) represents spherical surface of the radius $h\sqrt{2}$. The points inside the spherical surface represent the elastic state. If the material is in a plastic range then the point (**s**) is on the surface of the sphere. In the space $\{\sigma_i\}$ of principal stresses the Huber-Mises yield criterion represents a circular cylinder with an axis of unit vector with $(1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$ as the cosine directors.

In the space $\{\sigma_i\}$ of principal stresses the stress tensor and its isotropic or deviatoric part are described by three components so in this space can be treated as vectors.

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

$$\mathbf{p} = (p, p, p) \quad (21)$$

$$\mathbf{s} = (\sigma_1 - p, \sigma_2 - p, \sigma_3 - p) \quad (22)$$

where

$$\mathbf{p} = (\mathrm{tr}\,\boldsymbol{\sigma}/3)\,\mathbf{1} \quad (23)$$

The geometrical interpretation of an isotropic part of stress tensor is the trisector defined by the unit vector with $(1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$ as cosine directors. Since

σ = **s** + **p**, the deviatoric stress represents deviation of the stress **σ** from the axis of the cylinder, which is presented in Figure 3. A deviation of stress from the axis of the cylinder symmetry is the measure of material effort. This distance is

$$|\,s\,| = \sqrt{s_i s_i} = \sqrt{s_1^2 + s_2^2 + s_3^2} \tag{24}$$

and is equal to the radius of the Huber-Mises cylinder.

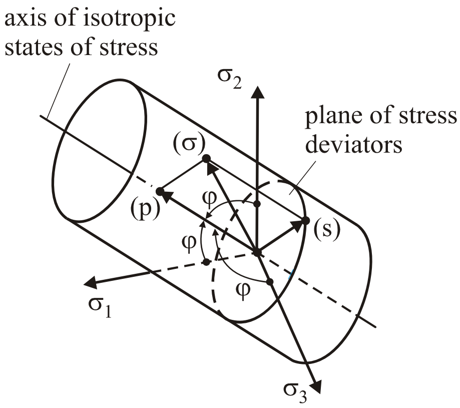

Fig. 3. The Huber-Mises yield locus in the space $\{\sigma_i\}$ of principal stresses

In the case of a plane state of strain the Huber-Mises yield criterion represents in the space $\{\sigma_{11}, \sigma_{22}, \sigma_{12}\}$ an elliptic cylinder with the axis on the plane $\{\sigma_{11}, \sigma_{22}\}$ defined by a unit vector with $\left(1/\sqrt{2},\ 1/\sqrt{2}\right)$ as cosine directors (Fig. 4).

In the case of a plane state of stress the Huber-Mises yield criterion in the space $\{\sigma_1, \sigma_2\}$ is represented by an ellipse being the trace of the cross section of the Huber-Mises cylinder by the plane $\sigma_3 = 0$ (Fig. 5).

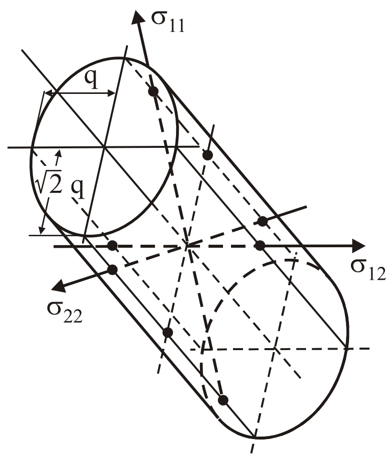

Fig. 4. The Huber-Mises yield locus for the plane state of strain in the space $\{\sigma_1 \times \sigma_{21} \times \sigma_{12}\}$

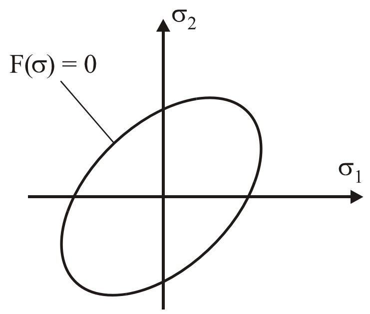

Fig. 5. The Huber-Mises yield locus for the plane state of stress in the space $\{\sigma_1 \times \sigma_2\}$

Based on Eq. (19) the Huber-Mises yield criterion can be written as

$$\left|\sigma_1 - \sigma_2\right|^n + \left|\sigma_2 - \sigma_3\right|^n + \left|\sigma_3 - \sigma_1\right|^n = 2\sigma_o^n \tag{25}$$

where n = 2. If $n \to \infty$ in Eq. (25) the yield criterion became the so-called the Treska yield criterion. According to the Treska criterion the loading function reads

$$F = \mathrm{Sup}_{i,j=1,2,3}\left(\sigma_i - \sigma_j\right) \tag{26}$$

The Treska yield criterion can be written as

$$\left[(\sigma_1 - \sigma_2)^2 - \sigma_o^2\right]\left[(\sigma_2 - \sigma_3)^2 - \sigma_o^2\right]\left[(\sigma_3 - \sigma_1)^2 - \sigma_o^2\right] = 0 \tag{27}$$

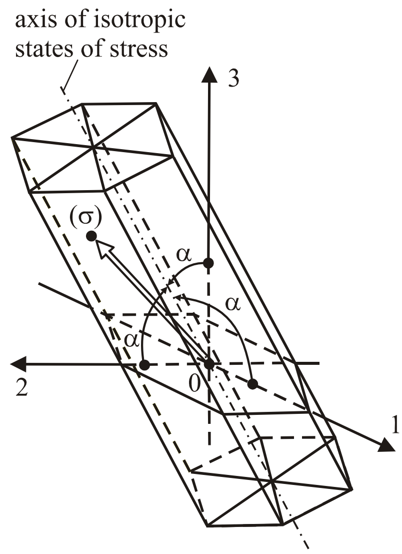

Fig. 6. The Treska yield criterion in the space $\{\sigma_i\}$

The geometrical interpretation of the Treska yield criterion is given in Figure 6. The Treska yield criterion for a plane state of stress is

$$(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2 = 4q^2 \tag{28}$$

It has the identical form as the Huber-Mises yield criterion (15) if we put $\sigma_{13} = \sigma_{23} = 0$ and $\sigma_3 = \frac{1}{2}(\sigma_{11} + \sigma_{22})$. The difference is when we change q onto σ_o. For the Treska yield criterion

$$\sigma_o = 2q \tag{29}$$

and for the Huber-Mises criterion

$$\sigma_o = \sqrt{3}q \tag{30}$$

The Treska yield criterion represents a prism inscribed in a Huber-Mises cylinder. Any plane orthogonal to the trisector, i.e. any deviatoric plane defined by σ = const intersects with the loading surface along a regular hexagon. A comparison of the Huber-Mises and the Treska yield criteria in the space $\{\sigma_i\}$ is given in Figure 7 and on the plane of deviators in Figure 8. On the plane $\sigma_3 = 0$ representing a plane state of stress the Huber-Mises and the Treska yield criteria are presented in Figure 9.

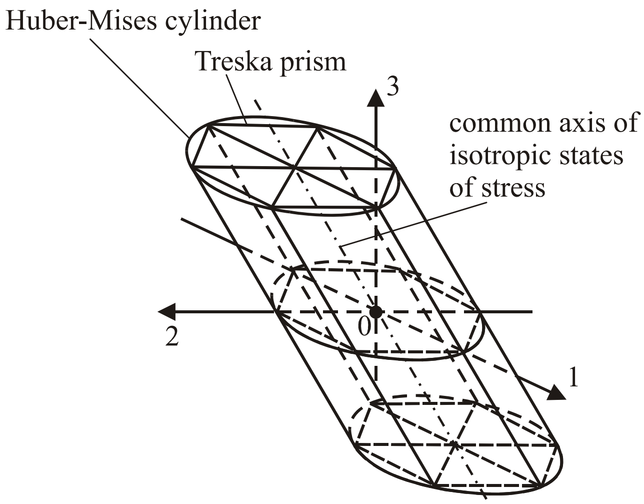

Fig. 7. Comparison of the Huber-Mises and the Treska yield criteria in the space $\{\sigma_i\}$

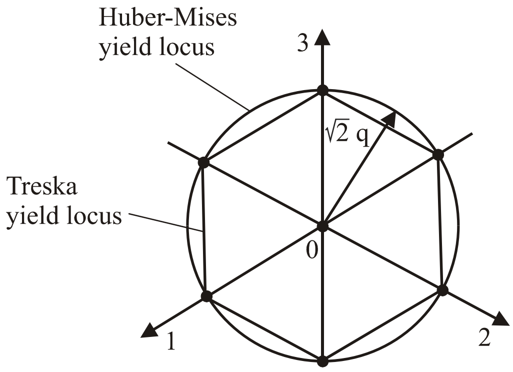

Fig. 8. Comparison of the Huber-Mises and the Treska yield criteria on a plane of deviators; plane normal to the cylinder and prism axis

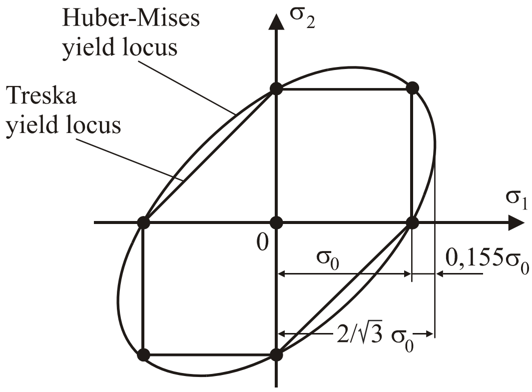

Fig. 9. Comparison of the Huber-Mises and the Treska yield criteria on a plane $\{\sigma_1, \sigma_2\}$

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Abstract

The purpose of the paper is to compare two basic yield criteria met in engineering mechanics i.e. Huber-Mises and Treska. The various forms of the yield locus are presented and discussed. The paper has a review character.

Streszczenie

Artykuł przedstawia analizę porównawczą dwóch kryteriów plastyczności, tj. Hubera-Misesa i Treski. Zaprezentowano różne postacie warunków plastyczności. Praca ma charakter przeglądowy.